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Quasidistributions for noncommuting cosine and sine operators

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Abstract. A similarity between the photon annihilation and creation operators and the Susskind-Glogower exponential phase operators motivates the introduction and study of quasidistributions of the cosine and sine of phase. These quasidistributions are related to the standard and antistandard orderings of trigonometric phase operators and to the normal, antinormal, and symmetrical orderings of the exponential phase operators. The symmetrical ordering is connected to an optical ideal tomography of the appropriate quasidistribution from the rotated Susskind-Glogower cosine operator.

PACS. 42.50.Dv Nonclassical field states; squeezed, antibunched, and sub-Poissonian states; operational definitions of the phase of the field; phase measurements

1 Introduction

The great importance of phase angle for classical optics does not cease to kindle the interest in the quantum phase. Since the very beginning of quantum theory the canonical conjugation between the action and angle variables of a harmonic oscillator has been reinterpreted as the number-phase canonical conjugation. Due to the difficulties with the Hermitian phase operator it has appeared that the probability operator-valued measure must be applied for the description of phase measurements [1,2].

A probability operator-valued measure as a resolution of the identity is useful for assigning probability distributions to statistical operators and also for assigning operators to classical physical quantities. The resolutions of the identity arising through the diagonalization of the so generated operators need not coincide. Suspectedly, they may differ from the original probability operator-valued measure. The identity resolution presented in [1,2] is usually called the Susskind-Glogower probability operator-valued measure. Susskind and Glogower [3] generated operators from the classical cosine and sine of phase and showed that the identity resolutions appropriate to the cosine and sine operators do not coincide and differ from the Susskind-Glogower probability operator-valued measure [4].

Independently, Garrison and Wong [5] considered an operator which could be generated from the classical phase angle whose range was $[-\pi, \pi)$. Here again another resolution of the identity was described different from those for cosine and sine operators and from the Susskind-Glogower probability operator-valued measure.

The phase-space methods of quantum theory have proved valuable in the phase-space approach to the quantum phase (for further details see the reviews [6–10]). The most important methods are connected to and distinguished by the orderings of the photon annihilation and creation operators and they provide at least but not only the identity resolutions which assign quasi-probability distributions to statistical operators and generate operators from classical physical quantities. It is possible to introduce not only the marginal phase-angle resolutions of the identity in this approach, but also the marginal action resolutions of the identity [11]. At present, the literature on the relationships with the experiment is so rich and devoted not only to the homodyne detection proper but also to the tomographical (reconstruction) methods that we cannot pay our attention to this interesting topic. Whereas the extraction of the phase information from the statistical operator is so straightforward that the comparison of the quasidistributions resulting from different phase approaches with the Susskind-Glogower probability distribution is immediate, the dual procedures of generating operators from classical phase quantities lead to analyses at different levels of intricacy. In the case of the antinormal ordering of annihilation and creation field operators, the exponential phase operators generated by the identity resolution have been introduced by Paul [12], whereas Turski [13] generated the phase-angle operator. The amount of effort and the depth of analysis comparable with the problems of Susskind and Glogower and Garrison and Wong can be found only in the work devoted to the Weyl quantization of phase angle, exponential of phase, and cosine and sine of phase [14–18].

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The statistical properties of a quantum state are actually those of the random variables whose realizations are the eigenvalues of measured operators. In quantum optics the only operator which can be measured directly is the number operator and the measurement consists in photocounting. Some detection schemes enable one to measure a quadrature operator and other more sophisticated ones accomplish simultaneous measurement of two quadratures. Thus the realistic measurements are described by the distribution of a quadrature as well as by a quasidistribution of two quadratures. The so-called realistic or operational approach to the quantum phase consists in deliberate confinement of the physicist to its properties which are based on the measured quadratures. Nevertheless, the success of the Hermitian phase operator proposed by Pegg and Barnett [19,20] evidences the interest in the ideal phase properties. Remarkably enough, the antinormal ordering of the photon annihilation and creation operators, to which the joint distribution of simultaneously measured quadratures in a familiar detection scheme [12,21] is related, has the antinormal ordering of the exponential phase operators as its counterpart in the ideal case [6,22,23]. If the antinormal ordering of the exponential phase operators is not considered, a violation of the trigonometric calculus occurs [24]. In that paper the quantum statistics of the Susskind-Glogower cosine operator have been evaluated for nonclassical states. It has been shown that for weak coherent signals the ideal cosine distribution is closer to the homodyne cosine distribution than the Susskind-Glogower cosine distribution can be. It has also been shown that for a coherent state with the average number of photons of three, the Susskind-Glogower cosine distribution approaches the ideal cosine distribution, whereas the homodyne cosine distribution is no more identical with any of them. In this paper we connect the desirable properties to the antinormal ordering, which can be used to describe the ideal and homodyne measurements, although different operators must be ordered.

In the sequel we will study the quasidistributions motivated by the analogy between the photon annihilation and creation operators and the Susskind-Glogower exponential phase operators. In Sections 2 and 3 we will expound the analogy between the quadrature operators and the cosine and sine of phase operators, which consists of a number of similarities both formal and of physical content. In Section 4 we will define four quasidistributions for quantum cosine and sine of phase, which are configured about the fifth, Wigner, quasidistribution for the Susskind-Glogower cosine and sine operators. Besides the special Wigner quasidistribution, we will pay a great attention to the quasidistributions related to the normal and antinormal orderings of the exponential phase operators and to the quasidistributions related to the standard and antistandard orderings of the cosine and sine operators. We will show the dichotomy that either the cosine and sine of phase obey the usual trigonometric relation or their quasidistribution is in a correspondence to a physical state. We will analyze the Wigner quasidistribution for the Susskind-Glogower cosine and sine operators in Section 5. We will show that the Laguerre polynomials and the Gaussian quasidistribution for the coherent state in the case of the usual Wigner function have analogues. These are particular Jacobi polynomials and a quasidistribution in a closed form.

2 Cosine and sine operators

The comprehensive text of Dirac [25] uses without hesitation the eigenstates of position coordinate and momentum observables. At present, the analysis of the optical homodyne tomography profits from the eigenstates of quadrature operators, although they are not elements of the Hilbert space whose complete basis are $|n\rangle$, $n = 0, 1, \ldots, \infty$, the familiar number states. The phase states [4]

$$|\varphi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \exp(in\varphi)|n\rangle \tag{2.1}$$

do not belong to this Hilbert space either. They are some of the eigenstates of the exponential phase operator,

$$\hat{E}_{-}|\varphi\rangle = \exp(i\varphi)|\varphi\rangle, \tag{2.2}$$

which is known to have the expansion

$$\hat{E}_{-} = \sum_{n=0}^{\infty} |n\rangle\langle n+1|, \tag{2.3}$$

where the subscript "−" indicates that this operator acts as a lowering operator. The expansion (2.3) is non-diagonal as well as that of the photon annihilation operator

$$\hat{a} = \sum_{n=0}^{\infty} \sqrt{n+1}|n\rangle\langle n+1|, \tag{2.4}$$

whereas the number operator $\hat{n} = \hat{a}^{\dagger}\hat{a}$ has the diagonal expansion in the basis of its eigenstates,

$$\hat{n} = \sum_{n=0}^{\infty} n|n\rangle\langle n|. \tag{2.5}$$

The photon creation operator $\hat{a}^{\dagger}$ has a similar expansion as the raising operator $\hat{E}_{+} = \hat{E}_{-}^{\dagger}$.

The phase states (2.1) have the property

$$\mathrm{Re}\,\langle\varphi|\varphi'\rangle = \frac{1}{4\pi} + \frac{1}{2}\delta(\varphi - \varphi') \quad \text{for } \varphi, \varphi' \in [\theta_0, \theta_0 + 2\pi), \tag{2.6}$$

where θ_0 is any real number, and enter the resolution of the identity,

$$\hat{1} = \int_{\theta_0}^{\theta_0+2\pi} |\varphi\rangle\langle\varphi|\, d\varphi. \tag{2.7}$$

The property (2.7) is sufficient for introducing phase probability densities

$$P_{\theta_0}(\varphi) = \mathrm{Tr}\left\{\hat{\rho}\hat{P}_{\theta_0}(\varphi)\right\}, \tag{2.8}$$

where $\hat{\rho}$ is the Hermitian operator describing the state of the system and $\hat{P}_{\theta_0}(\varphi)$ is the operator-valued density,

$$\hat{P}_{\theta_0}(\varphi) = \begin{cases} |\varphi\rangle\langle\varphi| & \text{for } \varphi \in [\theta_0, \theta_0 + 2\pi), \\ \hat{0} & \text{elsewhere.} \end{cases} \tag{2.9}$$

On substituting (2.9) into (2.8), the phase probability density is explicit

$$P_{\theta_0}(\varphi) = \begin{cases} \langle\varphi|\hat{\rho}|\varphi\rangle & \text{for } \varphi \in [\theta_0, \theta_0 + 2\pi), \\ 0 & \text{elsewhere.} \end{cases} \tag{2.10}$$

When one neglects the conditions in the definition (2.9), one obtains a 2π-periodic "density" from (2.8). The questions raised in this connection are treated in [26]. A few formalisms of quantum phase converge to the canonical phase probability density (2.8) [27,28].

Somewhat different situation occurs with respect to the operators related to functions of classical phase. Of course, the operators [4]

$$\hat{C} = \int_{\theta_0}^{\theta_0+2\pi} \cos\varphi \, |\varphi\rangle\langle\varphi| \, d\varphi,$$
$$\hat{S} = \int_{\theta_0}^{\theta_0+2\pi} \sin\varphi \, |\varphi\rangle\langle\varphi| \, d\varphi, \tag{2.11}$$

could be useful,

$$\hat{E}_- = \hat{C} + i\hat{S}, \tag{2.12}$$

but they do not commute and thus they cannot be simultaneously measured. Should other operators be generated then using the resolution of the identity? Maybe the complexity of their matrix in the number state basis decides. The Garrison-Wong phase operator [5] is too complicated in this respect, although it can be generated similarly as indicated by (2.11), but the sine and cosine operators are appropriate. When we take the exponential phase operator for related closely enough to the annihilation operator $\hat{a}$, in fact,

$$\hat{E}_- = (\hat{a}\hat{a}^\dagger)^{-\frac{1}{2}}\hat{a} = (\hat{n} + \hat{1})^{-\frac{1}{2}}\hat{a}, \tag{2.13}$$

and observe that the commutator

$$\left[\hat{E}_-, \hat{E}_+\right] = |0\rangle\langle 0| \tag{2.14}$$

is a projection operator, "just as" the identity operator, $[\hat{a}, \hat{a}^\dagger] = \hat{1}$, and that the commutator

$$\left[\hat{C}, \hat{S}\right] = \frac{i}{2}|0\rangle\langle 0| \tag{2.15}$$

is similar to $[\mathrm{Re}\,\hat{a}, \mathrm{Im}\,\hat{a}] = (i/2)\hat{1}$, we have a basis for our study. We accept the property (2.14) with a great deal of optimism, although Pegg and Barnett started their work on a new formalism with criticism of this property [19,20]. Traditionally, the attention has been focused on the commutators which correspond better to the Poisson brackets [29]

$$\left[\hat{n}, \hat{S}\right] = i\hat{C}, \quad \left[\hat{n}, \hat{C}\right] = -i\hat{S}. \tag{2.16}$$

This property is shared by the operators $\mathrm{Re}\,\hat{a}$, $\mathrm{Im}\,\hat{a}$,

$$[\hat{n}, \mathrm{Im}\,\hat{a}] = i\mathrm{Re}\,\hat{a}, \quad [\hat{n}, \mathrm{Re}\,\hat{a}] = -i\mathrm{Im}\,\hat{a}, \tag{2.17}$$

while the cosine and sine operators are distinguished by their spectra filling the interval $(-1, 1)$.

The well-known coherent states $|\alpha\rangle$ are characterized by the relation $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. Almost equally well-known coherent phase states $|\rho e^{i\varphi}\rangle$, $0 \le \rho < 1$, are defined by an analogous relation

$$\hat{E}_-|\rho e^{i\varphi}\rangle = \rho e^{i\varphi}|\rho e^{i\varphi}\rangle. \tag{2.18}$$

A rather peculiar definition as eigenstates of non-Hermitian operators $\hat{a}$, $\hat{E}_-$ is illuminated by the relationship to the original and generalized Heisenberg-Robertson uncertainty relations

$$\langle(\Delta\mathrm{Re}\,\hat{a})^2\rangle\langle(\Delta\mathrm{Im}\,\hat{a})^2\rangle \ge \frac{1}{16}, \tag{2.19}$$

$$\langle(\Delta\hat{C})^2\rangle\langle(\Delta\hat{S})^2\rangle \ge \frac{1}{16}\,[p(0)]^2, \tag{2.20}$$

where $\Delta\hat{x} = \hat{x} - \langle\hat{x}\rangle$, any operator $\hat{x}$, and $p(0)$ is the probability that the measured photon number is zero. In our case the Heisenberg uncertainty relation is very optimistic, because it involves no lower bound on the uncertainty product for the states, where the measured photon number cannot be zero.

The feature (2.15) of quantum world becomes clear cut when formulated classically using the Poisson bracket. Normally, the attention is paid to the relation

$$\{\cos\varphi, \sin\varphi\} = 0, \tag{2.21}$$

whence the interest follows in the Pegg-Barnett formalism [19,20], where the quantization of this Poisson bracket is accomplished, of course, with respect to canonical quantization. The Susskind-Glogower operators, however, can correspond rather to the classical quantities

$$C = \Theta(J)\cos\varphi, \quad S = \Theta(J)\sin\varphi, \tag{2.22}$$

where $\Theta(x)$ is the (Heaviside) unit-step function. The intensity-phase conjugation reads as

$$\{J, \phi\} = 1. \tag{2.23}$$

Then,

$$\{C, S\} = \delta(J)\Theta(J), \tag{2.24}$$

where $\delta(x)$ is the Dirac delta function.

Not only $\{C, S\} \neq 0$, but its exact value passes well the comparison with the quantum commutator (2.15), if the product of the generalized functions is appropriately defined. To this end we regularize $\delta(J)$ using a sequence of even functions. In the limit we obtain that

$$\delta(J)\Theta(J) = \frac{1}{2}\delta(J). \tag{2.25}$$

Hence,

$$\{C, S\} = \frac{1}{2}\delta(J). \tag{2.26}$$

Because the canonical quantization is accomplished with a correspondence

$$\{C, S\} \leftrightarrow -\frac{i}{\hbar}[\hat{C}, \hat{S}], \tag{2.27}$$

we observe that we rely on the correspondence

$$\delta(J) \leftrightarrow \frac{1}{\hbar}|0\rangle\langle 0|, \tag{2.28}$$

which is acceptable. It is easy to see that the quantities C and S have the usual Poisson brackets,

$$\{J, S\} = C, \;\; \{J, C\} = -S. \tag{2.29}$$

Let us remark that a more detailed inspection can relate the constant term in (2.6) to the photon vacuum. Pegg and Barnett and other authors have observed that the Susskind-Glogower formalism in Lévy-Leblond's scheme [30] is not then able to determine the statistical properties of the vacuum correctly [8].

3 Cosine and sine eigenstates

The homodyne detection can in principle measure oscillating distributions of a field quadrature. When a simultaneous measurement of two conjugate quadratures is attempted, the double homodyne detection can be realized using a 50:50 beamsplitter. Due to attenuation of the beamsplitter, the quadrature measurement suffers from a lower signal-to-noise ratio. In fact, fundamental analyses of the heterodyne and double heterodyne detections show that the quantum probability density oscillations of quantum origin are lost. Interestingly enough, the Susskind-Glogower cosine and sine probability densities can oscillate in some important case of the field, whereas the cosine and sine distributions determined via the formalism of canonical quantum phase [6] lack of these oscillations. The canonical phase measurement can be interpreted as a simultaneous measurement of the cosine and sine of phase.

In quantum mechanics, the operators whose quantum Poisson bracket is equal to unity are called canonically conjugate. Particularly, in the case of finite dimensional Hilbert spaces it is useful to speak of the canonical conjugation of observables, projection-valued measures or simply eigenkets. The position and momentum operators are canonically conjugate and, in consequence, their eigenkets are also canonically conjugate. Any position (coordinate) is equally probable in a momentum eigenstate to give an example. The question arises, whether the cosine and sine operators have a residual canonical conjugation property such that both cosines are equally probable in a sine eigenstate. The residual or restricted canonical conjugation means that we cannot speak of any cosines, but only of the two cosines obeying the well-known trigonometric identity, $C = \pm\sqrt{1 - S^2}$.

We will consider the trigonometric function operators

$$\hat{C} = \mathrm{Re}(\hat{E}_-), \;\; \hat{S} = \mathrm{Im}(\hat{E}_-) \tag{3.1}$$

in analogy to the quadrature operators, the position-like and the momentum-like one

$$\hat{Q} = 2\,\mathrm{Re}\,\hat{a}, \;\; \hat{P} = 2\,\mathrm{Im}\,\hat{a}, \tag{3.2}$$

respectively. The representation of eigenstates $|Q\rangle$ of $\hat{Q}$ in the number state basis,

$$|Q\rangle = \sum_{n=0}^{\infty} \langle n|Q\rangle |n\rangle, \tag{3.3}$$

where

$$\langle n|Q\rangle = \langle Q|n = 0\rangle \frac{1}{2^{\frac{n}{2}}\sqrt{n!}} H_n\left(Q/\sqrt{2}\right), \tag{3.4}$$

comprises the Hermite polynomials ($x = Q/\sqrt{2}$),

$$H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2), \tag{3.5}$$

known as orthogonal polynomials with respect to the Gaussian weight

$$\langle Q|n = 0\rangle^2 = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{Q^2}{2}\right). \tag{3.6}$$

Here the orthogonal polynomials combine with the probability density of a quadrature in the vacuum state. Let us remember that the coefficients in the expansion (3.3) are equal to the wave functions for the number states

$$\langle Q|n\rangle = \langle n|Q\rangle. \tag{3.7}$$

The appropriate representation of eigenstate $|P\rangle$ of $\hat{P}$ is derived using the property

$$|P\rangle = i^{\hat{n}}|Q = P\rangle. \tag{3.8}$$

Let us note that the unitary operator $i^{\hat{n}} = \exp(i\hat{n}\pi/2)$ represents the rotation of the phase space by $\pi/2$.

The states $|Q\rangle$, $|P\rangle$ have the orthogonality property,

$$\langle Q|Q'\rangle = \delta(Q - Q'), \;\; \langle P|P'\rangle = \delta(P - P'). \tag{3.9}$$

From (3.9) it follows that the quadrature eigenstates cannot be normalized. In fact, we encounter the projection-valued densities here. By the relation (3.2) the operators

$\hat{Q}$ and $\mathrm{Re}\,\hat{a}$ are distinct, and so are the systems of the eigenvectors. To compare them, one does not only make the substitution of the eigenvalue, but also multiplies the density $|Q\rangle\langle Q|$ by two, to respect a higher concentration of the eigenstates of the projection operator $\mathrm{Re}\,\hat{a}$.

A quadrature distribution can be obtained experimentally by means of the homodyne detection [31]. In principle, this distribution can have oscillations of quantum origin. The probability density of the eigenvalue of the quadrature operator $\hat{Q}$ in the number state $|n\rangle$,

$$\Phi_{\mathcal{S}}^{Q}(Q|n) = \langle Q|n\rangle^2, \tag{3.10}$$

has such oscillations. Whereas the formula (3.10) describes a separate measurement of a quadrature, a simultaneous measurement of canonically conjugate quadratures leads to the probability density [32,33]

$$\Phi_{\mathcal{A}}^{Q}(Q|n) = \frac{1}{2}\int_{-\infty}^{\infty} \Phi_{\mathcal{A}}^{QP}\left(\frac{Q}{2} + iy\middle|n\right) dy, \tag{3.11}$$

where the quasidistribution related to the antinormal ordering of the annihilation and creation operators

$$\Phi_{\mathcal{A}}^{QP}(\alpha|n) = \frac{1}{\pi n!}|\alpha|^{2n}\exp(-|\alpha|^2), \tag{3.12}$$

which becomes

$$\Phi_{\mathcal{A}}^{Q}(Q|n) = \frac{1}{\sqrt{\pi}}(-1)^n \frac{1}{2^{2n+1}} \times \exp\left(-\frac{Q^2}{4}\right) L_n^{-n-\frac{1}{2}}(Q^2), \tag{3.13}$$

with the Laguerre polynomials,

$$L_n^{\gamma}(x) = \Gamma(n+\gamma+1) \times \sum_{j=0}^{n} \frac{(-x)^j}{j!(n-j)!\Gamma(j+\gamma+1)}. \tag{3.14}$$

The comparison of the distributions of a quadrature in the number states resulting from the measurement of the appropriate operator and from the procedure of the simultaneous measurement of this operator and the appropriate conjugate one can be accomplished according to Figures 1 and 2. The probability densities $\Phi_{\mathcal{S}}^{Q}(Q|n)$ in Figure 1 exhibit quantum oscillations between two quasiclassical maxima tracing two arcs of a parabola. In the case of the probability densities $\Phi_{\mathcal{A}}^{Q}(Q|n)$, the oscillatory behaviour is smoothed out, as can be seen in Figure 2, by comparison with Figure 1 keeping a fixed n. Taking into account the probabilistic meaning of (3.11) and the graphic properties of the quasidistribution (3.12), we may obtain a picture of the probability density $\Phi_{\mathcal{A}}^{Q}(Q|n)$ on the quasiclassical grounds. The peaks of this distribution are situated at about $Q = 2\sqrt{n}$, for $|Q|$ larger the probability density of the quadrature is negligibly small and it is almost uniform between the peaks for n large enough ($n \gtrsim 15$).

In any case, when the diagonalization of the operator resorts to a continuous projection-valued density,

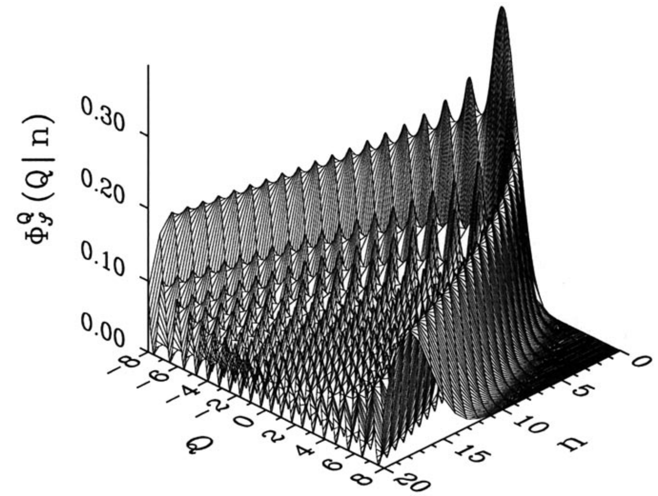

Fig. 1. The probability densities of the position-like quadrature in the number state $|n\rangle$ with $n = 0, 1, \ldots, 20$.

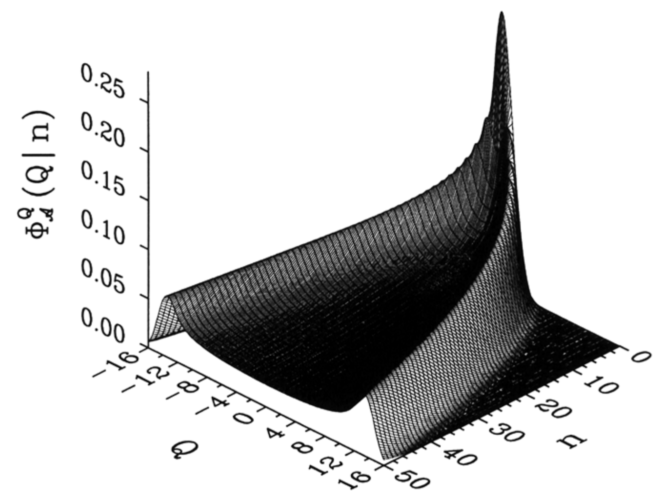

Fig. 2. The probability densities of the position-like quadrature in the number state $|n\rangle$ with $n = 0, 1, \ldots, 50$. A joint measurement of the position-like and momentum-like quadratures is assumed.

a substitution of the eigenvalue must be completed with a multiplication of the density by a positive factor. In the review article of Carruthers and Nieto [4], the states denoted as eigenstates of the cosine or sine operators are rather the eigenstates of $\cos^{-1}(\hat{C})$, $\sin^{-1}(\hat{S})$ according to their Dirac normalization, whereas in the communication of D'Ariano and Paris [24] the genuine eigenstates of the operator $\hat{C}$ are studied. The expansion of eigenstates $|C\rangle$ of $\hat{C}$ in terms of the number states,

$$|C\rangle = \sum_{n=0}^{\infty} \langle n|C\rangle |n\rangle, \tag{3.15}$$

where

$$\langle n|C\rangle = \sqrt{\frac{2}{\pi}}\sqrt[4]{1-C^2}\, U_n(C), \quad C \in [-1,1], \tag{3.16}$$

involves the Chebyshev polynomials of the second kind ($x = C$),

$$U_n(x) = \frac{\sin[(n+1)\cos^{-1} x]}{\sin(\cos^{-1} x)} \cdot \tag{3.17}$$

These are known to be the orthogonal polynomials with respect to the weight

$$\langle C|n=0\rangle^2 = \frac{2}{\pi}\sqrt{1-C^2}, \;\; C \in [-1,1]. \tag{3.18}$$

The coefficients in the series (3.15) are equal to the cosine representation for the number states

$$\langle C|n\rangle = \begin{cases} \langle n|C\rangle & \text{for } \; C \in [-1,1], \\ 0 & \text{for } \; C \,\overline{\in}\, [-1,1]. \end{cases} \tag{3.19}$$

In (3.15) we have not introduced $|C\rangle = 0$ for $C \,\overline{\in}\, [-1,1]$, the null vector, which does not describe a state of any physical system. The representation of the eigenstates $|S\rangle$ of $\hat{S}$ is derived using the property

$$|S\rangle = i^{\hat{n}}|C = S\rangle. \tag{3.20}$$

The states $|C\rangle$, $|S\rangle$ have the orthogonality property,

$$\langle C|C'\rangle = \delta(C - C'), \;\; \langle S|S'\rangle = \delta(S - S'), \tag{3.21}$$

where $C, C', S, S' \in (-1,1)$.

The philosophy of the Pegg-Barnett formalism expects the same phase distribution in all number states and it excludes that the cosine representation of a Fock state depends on the photon number. Nevertheless, the probability densities of the eigenvalue of the cosine operator $\hat{C}$ in the number state $|n\rangle$,

$$\Phi^C_{\mathcal{S}}(C|n) = \langle C|n\rangle^2, \tag{3.22}$$

are distinct. The canonical probability density of the eigenvalue of the cosine operator does not depend on the photon number,

$$\Phi^C_{\mathcal{A}}(C|n) = \begin{cases} \dfrac{1}{\pi}\dfrac{1}{\sqrt{1-C^2}} & \text{for } \; |C| < 1, \\ 0 & \text{for } \; |C| > 1. \end{cases} \tag{3.23}$$

Figure 3 demonstrates that the Susskind-Glogower probability density can oscillate about the canonical probability density. In the Pegg-Barnett system it is interesting that for $C \to \pm 1$ the probability density is large. In the case of the Susskind-Glogower manifestly quantal cosine probability density, this quasiclassical property is reflected by peaks starting with $n = 1$. We consider the Pegg-Barnett system more quasiclassical, because it enables cosine and sine to be measured simultaneously. Whereas the U-shape in the framework of the Pegg-Barnett formalism is of quasiclassical character, the oscillations are of a manifest quantum origin.

As the states $|Q\rangle$, $|P\rangle$, $|C\rangle$, $|S\rangle$ fall outside the Hilbert space, although we know they can at least be expanded in

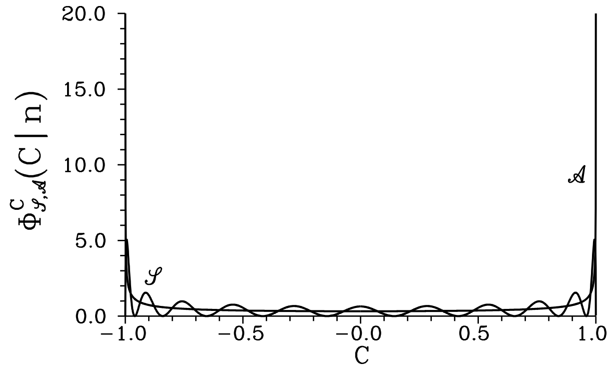

Fig. 3. The properties of the cosine probability density in the number state with $n = 10$: the Susskind-Glogower *versus* Pegg-Barnett prediction.

the number state basis, the computation with them can be pleasant and sure only with some ingredient of "physically preparable" states [20]. Interpreting the orthogonality relations (3.9, 3.21) for these states as the representations of the states $|Q'\rangle$, $|P'\rangle$, $|C'\rangle$, $|S'\rangle$ in the $|Q\rangle$-, $|P\rangle$-, $|C\rangle$-, $|S\rangle$-bases, respectively, we observe that the lack of a regular element on the left-hand sides leads to the generalized function on the right-hand sides.

The scalar product

$$\langle Q|P\rangle = \frac{1}{2\sqrt{\pi}}\exp\left(i\frac{QP}{2}\right), \tag{3.24}$$

which may also be interpreted as the wave function of the state $|P\rangle$, is a regular function, but it is not square integrable. A standard expression

$$|\langle Q|P\rangle|^2 = \frac{1}{4\pi} \tag{3.25}$$

does not yield a probability density of the quadrature Q in the state $|P\rangle$, although its constancy reflects the fact that the quadrature operators $\hat{Q}$, $\hat{P}$ are canonically conjugate. Of course, any suitable truncation of the Hilbert space $|Q\rangle$-basis leads to a replacement of (3.25) by a uniform probability distribution in a regularized state $|P\rangle_{reg}$.

To derive the analogue of (3.24), we use the relation

$$|C\rangle = \frac{1}{i\sqrt[4]{1-C^2}}\left[\exp(i\varphi)|\varphi\rangle - \exp(-i\varphi)|-\varphi\rangle\right], \tag{3.26}$$

where

$$\varphi = \cos^{-1} C, \;\; C \in [-1,1]. \tag{3.27}$$

In the course of computation, we use the relation (2.6) supplemented with the property

$$\mathrm{Im}\langle\varphi|\varphi'\rangle = -\frac{1}{4\pi}\mathcal{P}_\varphi\cot\left(\frac{\varphi-\varphi'}{2}\right). \tag{3.28}$$

Hence the phase representation of the phase state $|\varphi'\rangle$ has the imaginary part, which is also a generalized function. Here $\mathcal{P}_\varphi$ (principal value) means that this generalized function is the limit of the cotangent function, which has been replaced by zero in symmetrical neighbourhoods of $\varphi = \varphi'$ (mod 2π). It can be found easily that the representation of the sine state $|S\rangle$ in the cosine basis reads

$$\langle C|S\rangle = \sqrt[4]{1-C^2}\sqrt[4]{1-S^2}\Big[-\frac{1}{\pi}\mathcal{P}_C\frac{1}{C^2+S^2-1} + i\,\mathrm{sgn}(CS)\delta(C^2+S^2-1)\Big], \tag{3.29}$$

where $\mathcal{P}_C$ relates to the singularities $C = \pm\sqrt{1-S^2}$. Similarly as in (3.19), the representation of the sine state $|S\rangle$ in the cosine basis,

$$\langle C|S\rangle = 0 \ \text{ for } C \overline{\in} [-1,1],\ S \in [-1,1]. \tag{3.30}$$

The "unphysical features," in fact quantum oscillations, of the probability distribution of the Susskind-Glogower cosine operator for number states [24] can be compared with the shape of the probability distribution of the quadrature operator for these states. The peculiar quantum statistics are avoided adopting a joint measurement of the two quadratures similarly as the canonical phase probability distribution defines satisfactorily a joint distribution of quantum cosine and sine.

Regarding the relation (2.2), we have remarked that the phase states are not the only eigenstates of the exponential phase operator. In fact, the normalized eigenstates of the exponential phase operator can be obtained by a regularization,

$$|\rho e^{i\varphi}\rangle = \sqrt{1-\rho^2}\rho^{\hat{n}}|\varphi\rangle,\ \ 0 \le \rho < 1, \tag{3.31}$$

ensuring that the eigenvalue equation (2.18) is fulfilled. Hence, the coherent phase states are regularized phase states. The formula (3.31) is a good motivation for the limit expressions

$$\langle C|S\rangle = \lim_{q\nearrow 1}\langle C|q^{\hat{n}}|S\rangle, \tag{3.32}$$

$$= \lim_{q\nearrow 1}\left(\langle C|q^{\hat{n}}|S\rangle\right)_{appr}, \tag{3.33}$$

where

$$\left(\langle C|q^{\hat{n}}|S\rangle\right)_{appr} = -\frac{1}{\pi}\frac{\sqrt{\sin\varphi\cos\varphi'}}{\cos^2\varphi+\sin^2\varphi'-1+i(1-q^2)\cos\varphi\sin\varphi'}, \tag{3.34}$$

with φ connected to C by (3.27) and φ' connected to S by the relation

$$\varphi' = \sin^{-1}S. \tag{3.35}$$

After an easy algebraic manipulation and using the fact that

$$\lim_{q\nearrow 1}\frac{1}{\pi}\frac{(1-q^2)CS}{(C^2+S^2-1)^2+(1-q^2)^2(CS)^2} = \mathrm{sgn}(CS)\delta(C^2+S^2-1), \tag{3.36}$$

$$\lim_{q\nearrow 1}\frac{C^2+S^2-1}{(C^2+S^2-1)^2+(1-q^2)^2(CS)^2} = \mathcal{P}_C\frac{1}{C^2+S^2-1}, \tag{3.37}$$

we arrive again at the relation (3.29). Let us denote

$$w(C,C',z) = \langle C|z^{\hat{n}}|C'\rangle, \tag{3.38}$$

$$= \frac{2}{\pi}\frac{(1-z^2)\sqrt[4]{1-C^2}\sqrt[4]{1-C'^2}}{(1-z^2)^2-4z(1+z^2)CC'+4z^2(C^2+C'^2)}, \tag{3.39}$$

so that in (3.32)

$$\langle C|q^{\hat{n}}|S\rangle = w(C,S,iq). \tag{3.40}$$

The regularized expression

$$|S\rangle_{reg} = \frac{1}{\sqrt{w(S,S,q^2)}}q^{\hat{n}}|S\rangle \tag{3.41}$$

enables us to study a consolidation of the Pythagorean theorem in the probability density

$$\Phi_S^C(C|S,q) = |\langle C|S\rangle_{reg}|^2, \tag{3.42}$$

where

$$|\langle C|S\rangle_{reg}|^2 = \frac{|w(C,S,iq)|^2}{w(S,S,q^2)}\,. \tag{3.43}$$

In fact,

$$\lim_{q\nearrow 1}|\langle C|S\rangle_{reg}|^2 = \frac{1}{2}\delta(C-\sqrt{1-S^2}) + \frac{1}{2}\delta(C+\sqrt{1-S^2}), \tag{3.44}$$

$$= \sqrt{1-S^2}\delta(C^2+S^2-1). \tag{3.45}$$

The probability densities (3.42) are distributions of the quantum cosine in the Susskind-Glogower sine states, which are regularized by a $q^{\hat{n}}$ nonunitary transformation, which sends an improper phase state into a coherent phase state. To the extent that is possible to infer the properties of a Hilbert space vector from a distribution, we see that the sine state involves cosine components fulfilling the usual trigonometric relation. These components are two: $\pm\sqrt{1-S^2}$. Although the formula (3.44) is much more complicated than (3.25), there is a reminder of canonical conjugation. It holds that

$$\Phi_S^C\left(\sqrt{1-S^2}\Big|\,S,q\right) = \Phi_S^C\left(-\sqrt{1-S^2}\Big|\,S,q\right), \tag{3.46}$$

which suggests that the two cosine components are equally probable.

The behaviour of the probability density (3.42) for different values of q is pictorialized in Figures 4–6. For $S = \pm 1/\sqrt{2}$ we observe that the cosine distribution has two humps, which sharpen into two peaks at $C = \pm 1/\sqrt{2}$

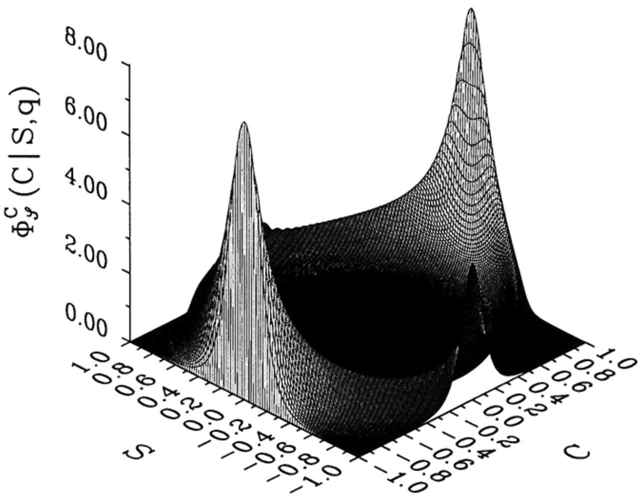

Fig. 4. The cosine probability density in the regularized eigenstate of the Susskind-Glogower sine operator for $S \in (-1, 1)$ and for the regularization parameter $q = 0.85$.

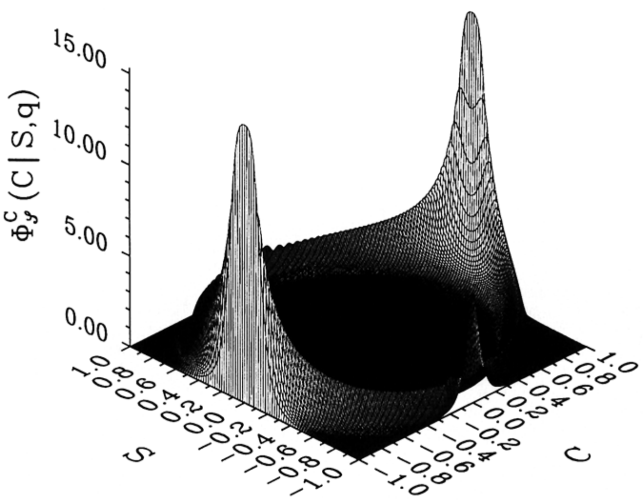

Fig. 5. The same as in Figure 4, but for $q = 0.9$.

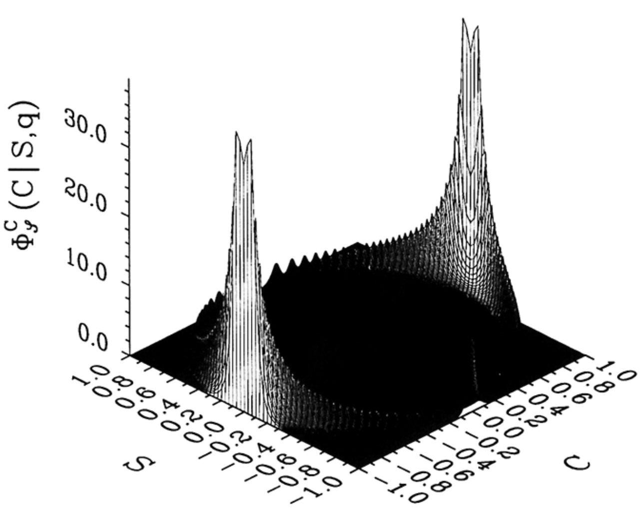

Fig. 6. The same as in Figure 4, but for $q = 0.95$.

with increasing q. The uniform value of q with respect to S does not mean equal regularization with respect to S as formalized by the formula

$$\Phi_{\mathcal{S}}^{C}(\pm\sqrt{1-S^2}|S, q) \simeq \frac{1}{\pi|S|(1-q^2)} \quad \text{for} \quad q \nearrow 1. \quad (3.47)$$

At least for $|S|$ small, we can see that the humps are sharpening. This may be connected with the effect of a barrier pressing for S close to zero. Although the probability density is not well-defined for $S = \pm 1$, we see a similar pressing effect about $C = 0$ for S tending to ± 1. From all the figures in the limit of q large, we see that the locus of the maxima of the probability density tends to be the circle $C^2 + S^2 = 1$.

4 Quasidistributions for quantum cosine and sine

Sometimes it is necessary to determine a joint $(k + l)$th-order moment of the quantities $\hat{A}$ and $\hat{B}$, where $\hat{A}$ and $\hat{B}$ are still arbitrary operators and k is the partial degree of $\hat{A}$ and l is the partial degree of $\hat{B}$. Fortunately, the physical problem itself usually determines the so-called operator ordering, which is not limited to the cases $\hat{A}^k\hat{B}^l$ or $\hat{B}^l\hat{A}^k$. For example, it is possible to consider the symmetrical expression $(1/2)(\hat{A}^k\hat{B}^l + \hat{B}^l\hat{A}^k)$, but the famous Weyl ordering cannot be characterized by such a simple formula, because it is based on the principle of a by far deeper symmetrization [34]. The literature is not very rich on the applications of the Weyl ordering to the operators different from the position coordinate and momentum operators or the quadrature operators. An observation of the Weyl ordering of the Susskind-Glogower cosine and sine operators is comprised in [35], where q-deformation has been considered. An attempt to analyze the Weyl correspondence in the case of number and momentum-like quadrature operators has been made [36].

In this section we intend to introduce the quasidistributions of eigenvalues of the cosine and sine operators. To this end we use the method of quantum characteristic function as outlined in [34].

The analogy between the operators $\hat{a}$, $\hat{a}^\dagger$, $(1/2)\hat{Q}$, $(1/2)\hat{P}$ on the one hand and the operators $\hat{E}_-$, $\hat{E}_+$, $\hat{C}$, $\hat{S}$ on the other hand has been so far going that it suggests the terms standard and antistandard orderings for the operators $\hat{C}$, $\hat{S}$ and the terms normal and antinormal orderings for the operators $\hat{E}_-$, $\hat{E}_+$, although they have been coined for the operators $(1/2)\hat{Q}$, $(1/2)\hat{P}$, $\hat{a}$, $\hat{a}^\dagger$ originally (*cf.* [37]).

Let us remark that there are also limitations to this analogy. The difference between the normal and antinormal orderings of the operators $\hat{a}$, $\hat{a}^\dagger$ is relatively small for the large-amplitude states, but it is absolutely present according to the commutation relation $[\hat{a}, \hat{a}^\dagger] = \hat{1}$. However, the difference between the normal and antinormal orderings of the operators $\hat{E}_-$, $\hat{E}_+$ is absolutely small for the large-amplitude states as follows from (2.14) and is indicated by equation (6.9) in [38].

To establish the notation and to illustrate the relationship of a quantum characteristic function to a quasidistribution, we assume that $\Phi^{CS}(C+iS)$ is a quasidistribution,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi^{CS}(C+iS)\,dC\,dS=1. \tag{4.1}$$

We introduce the corresponding characteristic function as

$$C^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)= \mathcal{F}\left[\Phi^{CS}(C+iS)\right]\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right), \tag{4.2}$$

where the Fourier transform

$$\mathcal{F}\left[\Phi^{CS}(C+iS)\right]\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(i\theta C+i\tau S)\Phi^{CS}(C+iS)\,dC\,dS. \tag{4.3}$$

The quasidistribution can be obtained as

$$\Phi^{CS}(C+iS)= \mathcal{F}^{-1}\left[C^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)\right](C+iS), \tag{4.4}$$

where

$$\mathcal{F}^{-1}\left[C^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)\right](C+iS) =\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(-i\theta C-i\tau S)C^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)d\theta\,d\tau. \tag{4.5}$$

The method of the quantum characteristic function is based on the use of the formulae (4.4, 4.5) as the defining relations. In principle, this is possible when the quasidistribution is given as an average (here a quantum average (4.12) below).

Let us address the quasidistributions related to the standard and antistandard orderings of the cosine and sine operators. The procedure of derivation of these quasidistributions is especially simple, but in connection with this the result is not very impressive, because the quasidistributions retain the phase factors in a somewhat inconvenient way. This procedure does not succeed in combining the phase factor of the cosine and sine representations, so that the resulting function could be real. Introducing the operators

$$\hat{D}_{st}^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)=\exp(i\theta\hat{C})\exp(i\tau\hat{S}), \tag{4.6}$$

$$\hat{D}_{antist}^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)=\exp(i\tau\hat{S})\exp(i\theta\hat{C}), \tag{4.7}$$

we may define the standard characteristic function

$$C_{st}^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)=\mathrm{Tr}\left\{\hat{\rho}\hat{D}_{st}^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)\right\} \tag{4.8}$$

and the antistandard characteristic function

$$C_{antist}^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)=\mathrm{Tr}\left\{\hat{\rho}\hat{D}_{antist}^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)\right\}, \tag{4.9}$$

where $\hat{\rho}$ is the statistical operator to be represented by the quasidistribution. Quite generally, we apply the prescription (4.4) in the domain of operators and we arrive at the operator-valued density

$$\hat{\Phi}^{CS}(C+iS)=\mathcal{F}^{-1}\left[\hat{D}^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)\right](C+iS). \tag{4.10}$$

Still generally, a subsequent use of (4.4) to the scheme

$$C^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)=\mathrm{Tr}\left\{\hat{\rho}\hat{D}^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)\right\} \tag{4.11}$$

leads to the rule

$$\Phi^{CS}(C+iS)=\mathrm{Tr}\left\{\hat{\rho}\hat{\Phi}^{CS}(C+iS)\right\}. \tag{4.12}$$

Now, the standard and antistandard orderings are pleasant, because [34,39]

$$\hat{\Phi}_{st}^{CS}(C+iS)=|C\rangle\langle C|S\rangle\langle S|, \tag{4.13}$$

$$=\langle C|S\rangle|C\rangle\langle S|\ \text{for}\ C,S\in[-1,1], \tag{4.14}$$

$$\hat{\Phi}_{antist}^{CS}(C+iS)=|S\rangle\langle S|C\rangle\langle C|, \tag{4.15}$$

$$=\langle S|C\rangle|S\rangle\langle C|\ \text{for}\ C,S\in[-1,1], \tag{4.16}$$

$$\hat{\Phi}_{st}^{CS}(C+iS)=\hat{\Phi}_{antist}^{CS}(C+iS)=\hat{0}\quad \text{for}\ C\ \text{or}\ S\,\overline{\in}\,[-1,1]. \tag{4.17}$$

From the formulae (4.14, 4.16) and the rule (4.12) it is obvious that these quasidistributions in a physically preparable state have the form

$$\Phi_{st}^{CS}(C+iS)=\langle C|S\rangle\langle S|\hat{\rho}|C\rangle, \tag{4.18}$$

$$\Phi_{antist}^{CS}(C+iS)=\langle S|C\rangle\langle C|\hat{\rho}|S\rangle; \tag{4.19}$$

here in the sense of (3.30) the regular functions $\langle S|\hat{\rho}|C\rangle$ and $\langle C|\hat{\rho}|S\rangle$ vanish for C or $S\,\overline{\in}\,[-1,1]$, and $\langle C|S\rangle$, $\langle S|C\rangle$ are generalized functions.

The reconstruction of the original statistical operator is possible according to the scheme

$$\hat{\rho}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi^{CS}(C+iS)\hat{\Delta}^{CS}(C+iS)\,dC\,dS. \tag{4.20}$$

In the case of the standard and antistandard orderings, we can use

$$\hat{\Delta}_{st}^{CS}(C+iS)=-\pi\frac{C^2+S^2-1}{\sqrt[4]{1-C^2}\sqrt[4]{1-S^2}}|S\rangle\langle C| \quad \text{for}\ C,S\in[-1,1], \tag{4.21}$$

$$\hat{\Delta}^{CS}_{antist}(C+iS) = -\pi \frac{C^2+S^2-1}{\sqrt[4]{1-C^2}\sqrt[4]{1-S^2}} |C\rangle\langle S| \quad \text{for}\, C,S \in [-1,1]. \quad (4.22)$$

The formulae (4.21, 4.22) comprise a regularizing factor, which makes unity with the scalar product (3.29). Because obviously (*cf.* (4.12, 4.17)),

$$\Phi^{CS}_{st}(C+iS) = \Phi^{CS}_{antist}(C+iS) = 0, \quad C \text{ or } S \,\overline{\in}\, [-1,1], \quad (4.23)$$

it is natural to take the integrals in (4.20) within the limits $-1, 1$ and the definitions (4.21, 4.22) are sufficient.

Profiting from the importance of the normal ordering of the annihilation and creation operators for bosons and fermions in quantum field theory and in quantum optics, we relate this term also to the exponential phase operators. We introduce the operator

$$\hat{D}^{CS}_{\mathcal{N}}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right) = \exp\left(\frac{-\tau+i\theta}{2}\hat{E}_+\right) \times \exp\left(\frac{\tau+i\theta}{2}\hat{E}_-\right). \quad (4.24)$$

Expanding the exponential functions in (4.24) in Taylor series and expressing the operator product $\hat{E}^k_+\hat{E}^l_-$ in the number state basis, we arrive at

$$\hat{D}^{CS}_{\mathcal{N}}\left(i\frac{\rho'}{2}e^{i\psi}\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \exp[i\psi(n-m)](-1)^{n-m} \times \sum_{k=0}^{m} \frac{(-1)^k}{k!\Gamma(k+n-m+1)}\left(\frac{\rho'}{2}\right)^{2k+n-m} |n\rangle\langle m|, \quad (4.25)$$

where

$$\psi = \arg(\theta+i\tau), \quad \rho' = \sqrt{\theta^2+\tau^2}. \quad (4.26)$$

From the rule (4.10), we obtain the operator-valued density

$$\hat{\Phi}^{CS}_{\mathcal{N}}(C+iS) = \frac{1}{\sqrt{1-C^2-S^2}} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{n!m!}\left[-\frac{\partial}{\partial(C-iS)}\right]^n \times \left[\psi - \frac{\partial}{\partial(C+iS)}\right]^m \delta(C+iS)|n\rangle\langle m| \quad (4.27)$$

or

$$\hat{\Phi}^{CS}_{\mathcal{N}}\left(\rho e^{i\varphi}\right) = \frac{1}{\pi\rho\sqrt{1-\rho^2}} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{(n+m)!} \times \exp[i(n-m)\varphi]\left(-\frac{\partial}{\partial\rho}\right)^{n+m} \delta(\rho)|n\rangle\langle m|. \quad (4.28)$$

Next we obtain the normal characteristic function according to (4.11) for any statistical operator $\hat{\rho}$. Particularly for the coherent phase state $|\overline{C}+i\overline{S}\rangle$, we get

$$C^{CS}_{\mathcal{N}}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\middle|\,\overline{C}+i\overline{S}\right) = \exp(i\theta\overline{C}+i\tau\overline{S}) \quad (4.29)$$

and from (4.4) we obtain

$$\Phi^{CS}_{\mathcal{N}}(C+iS|\overline{C}+i\overline{S}) = \delta(C+iS-\overline{C}-i\overline{S}), \quad (4.30)$$

where $\delta(z)$ is the Dirac delta function of the complex variable z. In general, taking into account the prescription (4.12), we arrive at the appropriate quasidistribution

$$\Phi^{CS}_{\mathcal{N}}(C+iS) = \frac{1}{\sqrt{1-C^2-S^2}} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \rho_{mn}\frac{1}{n!m!} \times \left[-\frac{\partial}{\partial(C-iS)}\right]^n \left[-\frac{\partial}{\partial(C+iS)}\right]^m \times\delta(C+iS) \quad (4.31)$$

$$= \frac{1}{\pi\rho\sqrt{1-\rho^2}} \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \rho_{mn}\frac{1}{(n+m)!} \times \exp[i(n-m)\varphi]\left(-\frac{\partial}{\partial\rho}\right)^{n+m} \delta(\rho). \quad (4.32)$$

It is easy to see that this quasidistribution enables one to obtain the diagonal representation of any statistical operator

$$\hat{\rho} = \int\int_{\overline{C}^2+\overline{S}^2<1} \Phi(\overline{C}+i\overline{S}) \times |\overline{C}+i\overline{S}\rangle\langle\overline{C}+i\overline{S}|\, d\overline{C}\, d\overline{S}, \quad (4.33)$$

where $\Phi(\overline{C}+i\overline{S})$ is a distribution confined to the interior of the unit disc. In this case, the normal characteristic function is given by (4.2), where $\Phi^{CS}(C+iS)$ is replaced with $\Phi(C+iS)$. It follows that the normal quasidistribution

$$\Phi^{CS}_{\mathcal{N}}(C+iS) = \begin{cases} \Phi(C+iS) & \text{for } C^2+S^2<1, \\ 0 & \text{elsewhere.} \end{cases} \quad (4.34)$$

Another consequence of the foregoing analysis is that

$$\hat{\Delta}^{CS}_{\mathcal{N}}(C+iS) = |C+iS\rangle\langle C+iS| \quad \text{for } C^2+S^2<1, \quad (4.35)$$

should be used in the reconstruction scheme (4.20). It is also natural to suit the integration domain of (4.20) to that of (4.33).

Let us dwell for a while on how to use the new quasidistributions to derive the usual phase probability density. Considering a 2π-periodic version of (2.10), we have

$$P(\varphi) = \langle\varphi|\hat{\rho}|\varphi\rangle. \quad (4.36)$$

Respecting the scheme (4.20), we obtain that

$$P(\varphi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi^{CS}(C+iS) \times \langle\varphi|\hat{\Delta}^{CS}(C+iS)|\varphi\rangle\, dC\, dS = \int_{\theta_0}^{\theta_0+2\pi}\int_0^{\infty} \Phi^{CS}(\overline{\rho}e^{i\overline{\varphi}}) \times \langle\varphi|\hat{\Delta}^{CS}(\overline{\rho}e^{i\overline{\varphi}})|\varphi\rangle\, \overline{\rho}\, d\overline{\rho}\, d\overline{\varphi}. \quad (4.37)$$

In the case of the normal ordering, the kernel of the integral expression (4.37) reads

$$\langle\varphi|\hat{\Delta}_{\mathcal{N}}^{CS}(\overline{\rho}e^{i\overline{\varphi}})|\varphi\rangle = \left|\langle\varphi|\overline{\rho}e^{i\overline{\varphi}}\rangle\right|^2 = \frac{1}{2\pi}\frac{1-\overline{\rho}^2}{1-2\overline{\rho}\cos(\varphi-\overline{\varphi})+\overline{\rho}^2}. \quad (4.38)$$

So the assumption that the state is a mixture of the coherent phase states (4.33) leads to a determination of the phase probability distribution as the mixture of the probability distributions (4.38).

In the framework of the normal ordering of the exponential phase operators, the undesirable properties of the Susskind-Glogower cosine and sine operators are illustrated with the convenient properties of the coherent phase states. Using these states, we always achieve the violation of the unit circle property (the usual trigonometric identity). In fact, the quasidistribution of the cosine and sine of the phase is supported by the whole unit disc, *cf.* (4.34). Although the mean photon number of the coherent phase states may achieve arbitrarily large values, so that these states may be arbitrarily far from the origin in a usual phase-space description, their normal and symmetrical quasidistributions are confined to the interior of the unit disc, when the ordering of the exponential phase operators is performed.

The antinormal ordering of the exponential phase operators has been repeatedly treated in the literature [6,22,23,27]. Since in this work there was no generating function for the antinormally ordered monomial operators, we introduce the operator

$$\hat{D}_{\mathcal{A}}^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right)=\exp\left(\frac{\tau+i\theta}{2}\hat{E}_-\right) \times\exp\left(\frac{-\tau+i\theta}{2}\hat{E}_+\right). \quad (4.39)$$

It holds that [6,40]

$$\hat{D}_{\mathcal{A}}^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right) = \int_{\theta_0}^{\theta_0+2\pi}\exp(i\theta\cos\varphi+i\tau\sin\varphi)|\varphi\rangle\langle\varphi|\,d\varphi. \quad (4.40)$$

Such Toeplitz operators (with respect to the number state basis) can be expanded in terms of the exponential phase operators

$$\hat{D}_{\mathcal{A}}^{CS}\left(i\frac{\rho'}{2}e^{i\psi}\right) = \sum_{k=-\infty}^{\infty}\exp(ik\psi)i^k J_k(\rho') \times\widehat{\exp}(-ik\varphi)\sum_{m=0}^{\infty}|m\rangle\langle m|, \quad (4.41)$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\exp[i\psi(n-m)] \times i^{n-m}J_{n-m}(\rho')|n\rangle\langle m|, \quad (4.42)$$

where ψ and ρ' are defined in (4.26). From the rule (4.10) in polar coordinates and using the orthogonality property of Bessel functions, we obtain the operator-valued density

$$\hat{\Phi}_{\mathcal{A}}^{CS}(C+iS) = 2\delta(C^2+S^2-1) \times|\varphi=\arg(C+iS)\rangle\langle\varphi=\arg(C+iS)| \quad (4.43)$$

or

$$\hat{\Phi}_{\mathcal{A}}^{CS}(\rho e^{i\varphi}) = 2\delta(\rho^2-1)|\varphi\rangle\langle\varphi|. \quad (4.44)$$

According to the formulae (4.12, 4.44), we obtain the quasidistribution for the antinormally ordered exponential phase operators

$$\Phi_{\mathcal{A}}^{CS}\left(\rho e^{i\varphi}\right) = 2\delta(\rho^2-1)\langle\varphi|\hat{\rho}|\varphi\rangle, \quad (4.45)$$

$$= 2\delta(\rho^2-1)P(\varphi). \quad (4.46)$$

Particularly, in the coherent phase state $|\overline{\rho}e^{i\overline{\varphi}}\rangle$,

$$\Phi_{\mathcal{A}}^{CS}\left(\rho e^{i\varphi}\middle|\,\overline{\rho}e^{i\overline{\varphi}}\right) = \frac{1}{\pi}\delta(\rho^2-1)\frac{1-\overline{\rho}^2}{1-2\overline{\rho}\cos(\varphi-\overline{\varphi})+\overline{\rho}^2}\,. \quad (4.47)$$

Using the properties of the Dirac delta function

$$\delta(\rho^2-1) = \frac{1}{2}[\delta(\rho-1)+\delta(\rho+1)],$$
$$\int_0^\infty \rho\delta(\rho+1)\,d\rho = 0, \quad (4.48)$$

we can verify that the generalized function (4.46) is a quasidistribution,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi_{\mathcal{A}}^{CS}(C+iS)\,dC\,dS = \int_{\theta_0}^{\theta_0+2\pi}\int_0^\infty \Phi_{\mathcal{A}}^{CS}\left(\rho e^{i\varphi}\right)\rho\,d\rho\,d\varphi = 1. \quad (4.49)$$

No embodiment of the scheme (4.20) exists, because the state of the physical system cannot be determined completely by the mere phase properties. For the large-amplitude states [38], the scheme (4.20) is not expected to operate well even for the normal and symmetrical orderings of the exponential phase operators.

5 Quantal-classical correspondence in the Wigner-Weyl sense

We were beyond the schemes according to [34] with the normal and antinormal orderings and only now we introduce the operator

$$\hat{D}_{\mathcal{S}}^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right) = \exp(i\theta\hat{C}+i\tau\hat{S}). \quad (5.1)$$

The generating function (5.1) of symmetrically ordered monomials in the operators $\hat{E}_-$, $\hat{E}_+$ admits a dynamical

$$\int_0^\infty i^{n-m} J_{n-m}(-\rho'\rho)\left[-i^{n+m+2}J_{n+m+2}(\rho')+i^{n-m}J_{n-m}(\rho')\right]\rho'\,d\rho' = \begin{cases} 2(n+1)\rho^{n-m}R_m^{(1,n-m)}(\rho^2) & \text{for } \rho<1, \\ 0 & \text{for } \rho>1, \end{cases} \tag{5.6}$$

$$\hat{\Phi}_S^{CS}(C+iS) = \frac{1}{\pi}\sum_{n=0}^{\infty}(n+1)\sum_{m=0}^{n}(C+iS)^{n-m}R_m^{(1,n-m)}(C^2+S^2)|n\rangle\langle m|$$

$$+\frac{1}{\pi}\sum_{m=0}^{\infty}(m+1)\sum_{n=0}^{m-1}(C-iS)^{m-n}R_n^{(1,m-n)}(C^2+S^2)|n\rangle\langle m| \;\text{ for }\; C^2+S^2<1, \tag{5.8}$$

$$\hat{\Phi}_S^{CS}(C+iS) = \hat{0} \;\text{ for }\; C^2+S^2>1. \tag{5.9}$$

$$\Phi_S^{CS}(C+iS) = \frac{1}{\pi}\sum_{n=0}^{\infty}(n+1)\sum_{m=0}^{n}\rho_{mn}(C+iS)^{n-m}R_m^{(1,n-m)}(C^2+S^2)$$

$$+\frac{1}{\pi}\sum_{m=0}^{\infty}(m+1)\sum_{n=0}^{m-1}\rho_{mn}(C-iS)^{m-n}R_n^{(1,m-n)}(C^2+S^2) \;\text{ for }\; C^2+S^2<1, \tag{5.10}$$

$$\Phi_S^{CS}(C+iS) = 0 \;\text{ for }\; C^2+S^2>1, \tag{5.11}$$

interpretation. This approach has been adopted in [41] and new states have been produced from the vacuum state using this unitary operator. Also the production of states from any given number state $|m\rangle$ has been described. The coefficients of the number-state representations are just the matrix elements of our operator generating function (5.1). The occurrence of the Bessel functions is reminiscent of the phase optimized states in [42]. In [41] it has been advertised that a suitable interaction could suppress a specific number-state component.

Using the property

$$\hat{D}_S^{CS}\left(-\frac{\tau}{2}+i\frac{\theta}{2}\right) = \exp(i\psi\hat{n})\hat{D}_S^{CS}\left(i\frac{\rho'}{2}\right)\exp(-i\psi\hat{n}), \tag{5.2}$$

where ψ and ρ' are defined in (4.26), we eliminate the sine operator from (5.1) and observe that

$$\hat{D}_S^{CS}\left(i\frac{\rho'}{2}\right) = \int_0^\pi \exp(i\rho'\cos\varphi)|\cos\varphi\rangle\langle\cos\varphi|\,d\varphi. \tag{5.3}$$

Here [4]

$$|\cos\varphi\rangle = \sqrt{\sin\varphi}\,|C=\cos\varphi\rangle = \sqrt{\frac{2}{\pi}}\sum_{n=0}^{\infty}\sin[(n+1)\varphi]|n\rangle. \tag{5.4}$$

Performing the indicated integration, we get the expansion in the number state basis

$$\hat{D}_S^{CS}\left(i\frac{\rho'}{2}e^{i\psi}\right) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\exp[i\psi(n-m)] \times\left[-i^{n+m+2}J_{n+m+2}(\rho')+i^{n-m}J_{n-m}(\rho')\right]|n\rangle\langle m|. \tag{5.5}$$

From the rule (4.10) using the formula

see equation (5.6) above

with $R_m^{(\alpha,\beta)}(x)$ the shifted Jacobi polynomials [43],

$$R_m^{(\alpha,\beta)}(x) = \frac{(-1)^m}{m!}(1-x)^{-\alpha}x^{-\beta} \times\frac{d^m}{dx^m}\left[(1-x)^{\alpha+m}x^{\beta+m}\right], \tag{5.7}$$

we obtain the operator-valued density

see equations (5.8, 5.9) above.

On substituting (5.8, 5.9) into (4.12), we obtain the Wigner function for the cosine and sine operators as

see equations (5.10, 5.11) above

where

$$\rho_{mn} = \langle m|\hat{\rho}|n\rangle. \tag{5.12}$$

From the formulae (4.11, 5.1, 5.2) we can see that the totality of characteristic functions of the rotated Susskind-Glogower cosine operators present the required quantum characteristic function in analogy with the reconstruction principle of Vogel and Risken [44]. Accordingly, we can develop the optical ideal tomography based on distributions of rotated Susskind-Glogower cosine operators. Ideally it is possible, we have in our mind the optical homodyne tomography, where the attribute homodyne is replaced by ideal, *i.e.*, we assume that instead of the rotated quadrature distributions the data on the rotated Susskind-Glogower cosine distributions should be acquired. Till now, no experiment is directly described by these operators. Interestingly enough, the reconstruction of the Susskind-Glogower cosine distribution has been published based, of course, on the rotated quadrature distributions [45]. Our distribution of the cosine of phase is confined in the interval $[-1, 1]$, whereas the Susskind-Glogower "cosine" phase distribution is supported by the interval $[0, \pi]$, it is just the distribution of the inverse cosine of the Susskind-Glogower cosine operator. Starting with this analogy, we can reveal the similarities as, for example, that the Gaussian formula for the Glauber coherent state has a Poisson-like formula for the coherent phase state as its counterpart. It may be interesting, because in spite of our obtaining this formula by algebraic manipulations, the original idea is based on the tomography and the numerical simulation can be carried out.

As an illustration, we have plotted the Wigner function for the cosine and sine operators in a coherent phase state and in a thermal state. The coherent phase state,

$$\hat{\rho}_{pcoh} = |\overline{\rho} e^{i\overline{\varphi}}\rangle\langle\overline{\rho} e^{i\overline{\varphi}}|, \tag{5.13}$$

with the ket defined in (3.31), is an ideal one in the single mode space, but it has the photon number distribution of a thermal state, which is a realistic one. We obtain this identity on equating $\overline{\rho}^2 = \overline{n}/(\overline{n}+1)$, where $\overline{n} = \mathrm{Tr}\{\hat{\rho}_{th}\hat{n}\}$, $\hat{\rho}_{th}$ being the thermal state. Using the formula for the generating function of the Jacobi polynomials [46], we obtain the Wigner function (5.10) for the coherent phase state

$$\begin{aligned} \Phi_{\mathcal{S}}^{CS}(C+iS) &\equiv \Phi_{\mathcal{S}}^{CS}\left(\rho\; e^{i\varphi} \,\middle|\, \overline{\rho} e^{i\overline{\varphi}}\right) \\ &= \frac{1-\overline{\rho}^2}{\pi} \frac{1}{\left[1+\overline{\rho}^2 - 2\overline{\rho}\rho\cos(\varphi-\overline{\varphi})\right]^2} \end{aligned} \tag{5.14}$$

and for the thermal state

$$\begin{aligned} \Phi_{\mathcal{S}}^{CS}(C+iS) &\equiv \Phi_{\mathcal{S}}^{CS}(C+iS|\overline{n}) \\ &= \frac{1}{4\pi} \frac{\left(\overline{n}+\frac{1}{2}\right)(\overline{n}+1)}{\left[\left(\overline{n}+\frac{1}{2}\right)^2 - \overline{n}(\overline{n}+1)(C^2+S^2)\right]^{\frac{3}{2}}} \\ &\quad \text{for } C^2+S^2 \le 1. \end{aligned} \tag{5.15}$$

The graph of (5.15, 5.11) in Figure 7 is rotationally invariant, because the thermal state lacks of the phase properties. The Wigner function attains its "edge" value when the variable $\rho = \sqrt{C^2+S^2}$ is unity. That is why the graph is cup like. The tendency to prefer the boundary is very strong, because although $\overline{n} = 1$, *i.e.*, the mean photon number equal to 1 is relatively small, the cup shape is conspicuous. To facilitate a comparison, we have chosen $\overline{n} = 1$, $\overline{\rho}^2 = 1/2$.

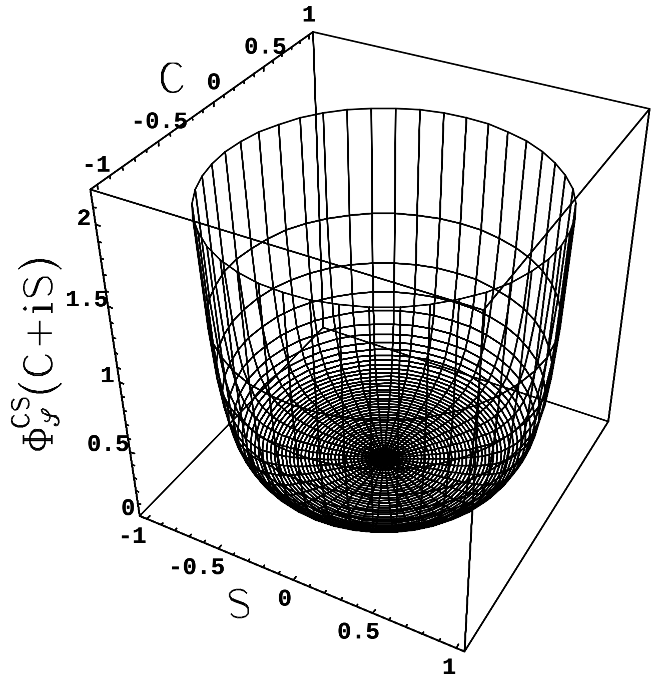

Fig. 7. The Wigner function for the cosine and sine operators in the thermal state with the mean photon number $\overline{n} = 1$.

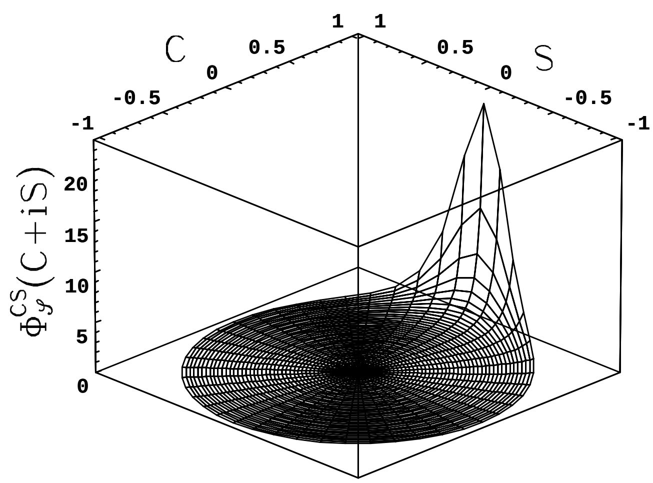

Fig. 8. The Wigner function for the cosine and sine operators in the coherent phase state with $\overline{\rho} = 1/\sqrt{2}$, $\overline{\varphi} = 0$.

The coherent phase state has the quasidistribution such that $\rho = 1$ is again the edge or peak value, but, moreover, this state has the preferred phase zero (*cf.* the peak at $C = 1$, $S = 0$) as illustrated in Figure 8. The fact that ρ is not certain to be unity, but that the quasidistribution has only the peak values at $\rho = 1$ is connected to the choice of the Susskind-Glogower cosine and sine operators and to their mimicking position coordinate and momentum operators with their obedience to the complementarity principle.

The illustrative examples have been performed with $\overline{n}$ small, $\overline{\rho}^2 \ll 1$, because it can be expected that for greater values the appropriate quasidistributions approach the antinormal ones according to (4.46) and the Dirac delta function cannot be plotted. The difference between the orderings consists in the near vacuum contribution, which is negligible for strong fields except some artificial examples of superposition states.

The contrast between the symmetrical ordering of the exponential phase operators and the symmetrical ordering of the photon annihilation and creation operators is very sharp. Returning to the case of thermal state, we can imagine the graph of the usual Wigner function, which is rotationally symmetrical, but the maximum is situated at the origin, $|\alpha| = 0$. The Wigner function of the coherent phase state with the preferred phase zero is elongated in the direction of the $\mathrm{Re}(\alpha)$ axis, but it has the maximum near the origin again. As stated above, the Wigner function for the cosine and sine operators attains rather a minimum near the origin.

Returning to the definitions (4.6, 4.7, 4.24, 4.39, 5.1), we find that the standard, antistandard and *symmetrical* orderings lead to quasidistributions yielding the cosine and sine distributions as the marginal ones (in the characteristic functions we may substitute $\tau = 0$, $\theta = 0$, respectively). In analogy to the quadrature case, the normal and antinormal orderings do not enjoy this property.

Let us describe the way in which the Laguerre polynomials enter the exposition of the usual quadrature Wigner function. This function is defined as

$$\Phi_{\mathcal{S}}^{QP}(\alpha) = \mathrm{Tr}\left\{\hat{\rho}\hat{\Phi}_{\mathcal{S}}^{QP}(\alpha)\right\}, \tag{5.16}$$

where the Wigner operator

$$\hat{\Phi}_{\mathcal{S}}^{QP}(\alpha) = \frac{2}{\pi}(-1)^{(\hat{a}^\dagger - \alpha^*\hat{1})(\hat{a}-\alpha\hat{1})}. \tag{5.17}$$

It is familiar that the inversion of (5.16) is possible,

$$\hat{\rho} = \int \Phi_{\mathcal{S}}^{QP}(\alpha)\hat{\Delta}_{\mathcal{S}}^{QP}(\alpha)\, d^2\alpha, \tag{5.18}$$

where the operators to be mixed

$$\hat{\Delta}_{\mathcal{S}}^{QP}(\alpha) = \pi\hat{\Phi}_{\mathcal{S}}^{QP}(\alpha). \tag{5.19}$$

Arbitrarily small thermalization of (5.19) leads to an operator whose trace exists and is unity.

Let us demonstrate that the useful properties of this formalism are connected with the orthogonality properties of the trigonometric and Laguerre polynomials. To this end, we invoke the mapping theorem

$$\mathrm{Tr}\{\hat{\rho}\hat{A}\} = \int \Phi_{\mathcal{S}}^{QP}(\alpha)A_{QP}^{(\mathcal{S})}(\alpha)\, d^2\alpha, \tag{5.20}$$

where

$$A_{QP}^{(\mathcal{S})}(\alpha) = \mathrm{Tr}\left\{\hat{\Delta}_{\mathcal{S}}^{QP}(\alpha)\hat{A}\right\}, \tag{5.21}$$

and we sketch a proof of its validity. In (5.20) we need not restrict ourselves to $\hat{\rho}$, which are statistical operators. The operators $\hat{\rho}$ and $\hat{A}$ can be expanded in the bases $\{|n\rangle\langle m|\}$, $\{|\overline{n}\rangle\langle\overline{m}|\}$, respectively. Now we put

$$\begin{aligned}\hat{\rho} &= |n\rangle\langle m|,\\ \hat{A} &= |\overline{n}\rangle\langle\overline{m}|,\end{aligned} \tag{5.22}$$

and we obtain from (5.20) that

$$\begin{aligned}I &\equiv \int \langle m|\hat{\Phi}_{\mathcal{S}}^{QP}(\alpha)|n\rangle\langle\overline{m}|\hat{\Delta}_{\mathcal{S}}^{QP}(\alpha)|\overline{n}\rangle\, d^2\alpha\\ &= \delta_{m\overline{n}}\delta_{\overline{m}n},\end{aligned} \tag{5.23}$$

where we have denoted the left-hand side by I for later use. Introducing the polar decomposition

$$\alpha = \frac{\sqrt{x}}{2}e^{i\varphi}, \tag{5.24}$$

taking into account that $d^2\alpha = (1/8)\, dx\, d\varphi$, the matrix element

$$\langle\overline{m}|\hat{\Delta}_{\mathcal{S}}^{QP}\left(\frac{\sqrt{x}}{2}e^{i\varphi}\right)|\overline{n}\rangle = \exp[i(\overline{m}-\overline{n})\varphi]\bar{l}_{\overline{n}\overline{m}}\left(\frac{\sqrt{x}}{2}\right), \tag{5.25}$$

where

$$\bar{l}_{\overline{n}\overline{m}}\left(\frac{\sqrt{x}}{2}\right) = \begin{cases} 2(-1)^{\overline{n}}\sqrt{\frac{\overline{n}!}{\overline{m}!}}(\sqrt{x})^{\overline{m}-\overline{n}}L_{\overline{n}}^{\overline{m}-\overline{n}}(x)\exp\left(-\frac{x}{2}\right) & \\ \qquad \text{for}\, \overline{m} \geq \overline{n}, & \\ \bar{l}_{\overline{m}\overline{n}}\left(\frac{\sqrt{x}}{2}\right) \qquad \text{for}\;\; \overline{m} \leq \overline{n}, & \end{cases} \tag{5.26}$$

and the similar expression for $\langle m|\hat{\Phi}_{\mathcal{S}}^{QP}(\sqrt{x}/2)|n\rangle$, we see that

$$I = \begin{cases} I_{m-n,\overline{m}-\overline{n}}\; I_{n,\overline{m}|m-n}^{QP} & \text{for}\;\; m \geq n,\\ I_{m-n,\overline{m}-\overline{n}}\; I_{m,\overline{n}|n-m}^{QP} & \text{for}\;\; m \leq n. \end{cases} \tag{5.27}$$

Here the integrals

$$\begin{aligned}I_{m-n,\overline{m}-\overline{n}} &= \frac{1}{2\pi}\int_0^{2\pi}\exp[i(m-n+\overline{m}-\overline{n})\varphi]\, d\varphi\\ &= \delta_{m-n,\overline{n}-\overline{m}}\end{aligned} \tag{5.28}$$

and

$$\begin{aligned}I_{n,\overline{m}|m-n}^{QP} &= (-1)^{n-\overline{m}}\sqrt{\frac{n!\overline{m}!}{m!\overline{n}!}}\int_0^\infty x^{m-n}e^{-x}\\ &\quad\times L_n^{m-n}(x)L_{\overline{m}}^{m-n}(x)\, dx = \delta_{\overline{m}n}.\end{aligned} \tag{5.29}$$

Similarly,

$$I_{m,\overline{n}|n-m}^{QP} = \delta_{m\overline{n}}. \tag{5.30}$$

Quite generally, the operators $\hat{\rho}$ and $\hat{A}$ are algebraic sums of the transition operators (5.22), so the mapping theorem (5.20) is proven.

Now we verify that the embodiment of the scheme (4.20) has the form

$$\hat{\rho} = \int\int_{C^2+S^2\leq 1} \Phi_{\mathcal{S}}^{CS}(C+iS) \times \hat{\Delta}_{\mathcal{S}}^{CS}(C+iS)\,dC\,dS, \quad (5.31)$$

where the operator

$$\hat{\Delta}_{\mathcal{S}}^{CS}(C+iS) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} u_{nm}(C^2+S^2) \times \langle n|\hat{\Phi}_{\mathcal{S}}^{CS}(C+iS)|m\rangle|n\rangle\langle m|, \quad (5.32)$$

with

$$u_{nm}(C^2+S^2) = \pi\frac{(n+m+1)}{(n+1)(m+1)}(1-C^2-S^2). \quad (5.33)$$

Equivalently, we verify the mapping theorem

$$\mathrm{Tr}\{\hat{\rho}\hat{A}\} = \int\int_{C^2+S^2\leq 1} \Phi_{\mathcal{S}}^{CS}(C+iS) \times A_{CS}^{(\mathcal{S})}(C+iS)\,dC\,dS, \quad (5.34)$$

where

$$A_{CS}^{(\mathcal{S})}(C+iS) = \mathrm{Tr}\left\{\hat{\Delta}_{\mathcal{S}}^{CS}(C+iS)\hat{A}\right\}. \quad (5.35)$$

Using the assumption (5.22), we obtain from (5.34) that

$$\begin{aligned} I \equiv \int\int_{C^2+S^2\leq 1} &\langle m|\hat{\Phi}_{\mathcal{S}}^{CS}(C+iS)|n\rangle \\ &\times \langle\overline{m}|\hat{\Delta}_{\mathcal{S}}^{CS}(C+iS)|\overline{n}\rangle\,dC\,dS \\ &= \delta_{m\overline{n}}\delta_{\overline{m}n}. \end{aligned} \quad (5.36)$$

Performing the substitution

$$\begin{aligned} C &= \sqrt{x}\ \cos\varphi \\ S &= \sqrt{x}\ \sin\varphi, \end{aligned} \quad (5.37)$$

taking into account that $dC\,dS = (1/2)\,dx\,d\varphi$, the matrix elements

$$\langle m|\hat{\Phi}_{\mathcal{S}}^{CS}(\sqrt{x}\cos\varphi, \sqrt{x}\sin\varphi)|n\rangle = \exp[i(m-n)\varphi]\overline{r}_{nm}(\sqrt{x}), \quad (5.38)$$

where

$$\overline{r}_{nm}(\sqrt{x}) = \begin{cases} \frac{1}{\pi}(m+1)(\sqrt{x})^{m-n}R_n^{(1,m-n)}(x) & \text{for } m\geq n, \\ \overline{r}_{mn}(\sqrt{x}) & \text{for } m\leq n, \end{cases} \quad (5.39)$$

and the appropriate expression for $\langle\overline{m}|\hat{\Delta}_{\mathcal{S}}^{CS}(C,S)|\overline{n}\rangle$, we verify that

$$I = \begin{cases} I_{m-n,\overline{m}-\overline{n}}\ I_{n,\overline{m}|m-n}^{CS} & \text{for } m\geq n, \\ I_{m-n,\overline{m}-\overline{n}}\ I_{m,\overline{n}|n-m}^{CS} & \text{for } m\leq n. \end{cases} \quad (5.40)$$

Here the integral

$$\begin{aligned} I_{n,\overline{m}|m-n}^{CS} &= \frac{(m+1)(\overline{n}+\overline{m}+2)}{(\overline{m}+1)}\int_0^1 (1-x)x^{m-n} \\ &\quad\times R_n^{(1,m-n)}(x)R_{\overline{m}}^{(1,m-n)}(x)\,dx \\ &= \delta_{\overline{m}n}. \end{aligned} \quad (5.41)$$

6 Conclusion

We have demonstrated that the five orderings of quadrature operators have their analogues in five orderings of the Susskind-Glogower cosine and sine operators. We have introduced the quasidistributions of eigenvalues of the cosine and sine operators using the method of ordered quantum characteristic function. Contrary to the usual situations, also the quasidistributions related to the standard and antistandard orderings are not regular but generalized functions. The support of the quasidistributions related to these orderings is the unit square and they can take on imaginary values. Except the antinormal ordering, all kinds of ordering violate the familiar relation between cosine and sine of the same angle. Accordingly, the support of the quasidistributions related to the normal and symmetrical orderings is the unit disc and the support for the antinormal ordering is the unit circle. The normal ordering of the exponential phase operators imitates that of the photon annihilation and creation operators up to the role played by the coherent phase state. A similar role of the coherent phase state in the antinormal ordering of the exponential phase operators is absent. The Weyl ordering of the cosine and sine operators leads to a quasidistribution whose connection to the number-state basis matrix elements is mediated by the analogues of the Laguerre polynomials similarly as the relationship between the cosine representation and the coefficients in the Fock state basis comprises relatives of the Hermite polynomials. The quasidistributions related to all but the antinormal ordering are in correspondence to a physical state. At the time that the Susskind-Glogower cosine and sine operators were scorned, because they were not simultaneously measurable, nobody took into account that at present the repeated measurement would make the optical homodyne tomography feasible. If we consider the Susskind-Glogower cosine operator instead of the quadrature operator in the homodyne measurement, we obtain another view of the quantum phase information retrievable *via* the Susskind-Glogower cosine and sine operators.

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